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Thick Lévy plates re-visited

K.H. Lee [a], G.T. Lim [b], C.M. Wang [b,*]

[a] *Department of Mechanical and Production Engineering, The National University of Singapore, Kent Ridge, Singapore 119260, Singapore*

[b] *Department of Civil Engineering, The National University of Singapore, Kent Ridge, Singapore 119260, Singapore*

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Abstract

This paper is concerned with the bending problem of Lévy plates which are simply supported on two opposite edges with any combination of simply supported, clamped or free conditions at the remaining two edges. This study attempts to solve thick Lévy plate problems in a novel way by establishing bending relationships that allow the prediction of Mindlin plate results using the corresponding Kirchhoff solutions. Based on the concept of load equivalence, these relationships obviate the need for complicated thick plate analyses that involve significant computation time and effort. Numerical plate solutions are then determined from these relationships and the validity of these results is verified using other known results and those generated using the ABAQUS software. It is through this study that the only analytical Mindlin plate solutions by Cooke and Levinson (Int. J. Mech. Sci. 25 (1983) 207) are found to contain errors. In this study, it is found that there are important distinctions between the Mindlin and Reissner plate theories. These differences will also be substantiated by numerical comparison. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Bending; Thick plates; Lévy solutions; Mindlin plate theory; Reissner plate theory

1. Introduction

In the analysis of relatively thin plates, it is sufficient to use the Kirchhoff plate theory. This classical plate theory can produce fairly accurate plate solutions until the thickness-to-length ratio reaches the limit of 1/20 (Yuan and Miller, 1992; Reddy, 1999). Once over this limit, the errors become significant enough to warrant the use of thick plate analysis. These errors arise because of the Kirchhoff plate assumption that the normals to the middle surface are to remain straight and normal after the plate has undergone deformation. This normality assumption is equivalent to the negligence of the effect of transverse shear deformation.

For moderately thick plates with thickness-to-length ratios over 1/20, it is necessary to adopt a plate theory that accounts for the effect of the transverse shear deformation. Many shear deformable plate theories, mostly of higher order, have been proposed over the years for the analysis of thick plates (Nelson

* Corresponding author. Tel.: +65-874-2157; fax: +65-779-1635.
E-mail address: cvewcm@nus.edu.sg (C.M. Wang).

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and Lorch, 1974; Reissner, 1975; Lo et al., 1977; Levinson, 1980; Reddy, 1984). These higher-order plate theories utilise displacement fields that truncate terms of different orders in the power series expansion through the plate thickness. With higher-order terms in the displacement field, the bending behaviour of thick plates can be predicted more accurately. However, it is noted that higher accuracy often comes with a price of longer computation time and effort.

As an alternative, one can use a simple shear deformable plate theory that adopts a displacement field of the same order as the classical thin plate theory, known as the Mindlin plate theory (Mindlin, 1951). Commonly referred to as a first-order shear deformable plate theory, the Mindlin plate theory primarily relaxes the Kirchhoff's normality assumption by allowing the normals to undergo uniform rotation about the middle surface after deformation. In allowing the normals to rotate uniformly, the Mindlin plate theory essentially assumes constant shear strains (and stresses) across the plate thickness. This however violates the statical requirement that shear stresses are to vanish at the free surfaces of the plate. To compensate for such an error, Mindlin introduced a shear correction factor to modify the shear modulus.

Research on thick plate analysis based on the Mindlin plate theory has mostly been restricted to numerical techniques for the generation of results. For Lévy plates, researchers employ numerical methods like the finite element method (Huang and Hinton, 1984; Bergan and Wang, 1984; Hinton and Huang, 1986; Yuan and Miller, 1992; Dong et al., 1993; Dong and Teixeira de Freitas, 1994), the finite strip method (Petrolito, 1990), the differential quadrature element method (Liu and Liew, 1998) or the segmentation method (Kant and Hinton, 1980) to produce Mindlin plate solutions. In the open literature, the only analytical Mindlin plate results on Lévy plates have been reported by Cooke and Levinson (1983). This has been found to be erroneous in Wang et al. (1999) and will be established in the present study. The other exact plate solutions are provided by Salerno and Goldberg (1960) for Lévy plates which is however based on another shear deformable plate theory, the Reissner plate theory. In general, plate solutions obtained by Kant and Hinton (1980) have been considered as the benchmark for comparison and validation. In recent years, Wang and his coworkers (Wang and Alwis, 1995; Wang and Lee, 1996; Wang, 1997; Wang et al., 1999; Wang and Lim, 1999) have initiated a new research direction on thick plate analysis by interestingly linking up the Kirchhoff plate solutions with the Mindlin counterparts in a single bending relationship. Via the concept of load equivalence, these novel relationships allow researchers and engineers to make use of the widely available Kirchhoff plate solutions (Timoshenko and Woinowsky-Krieger, 1959; Mansfield, 1989) to bypass complicated thick plate analyses. So far, bending relationships using this method have been derived for circular and annular plates, sectorial plates and polygonal plates.

Although similar Kirchhoff–Mindlin bending relationships for Lévy plates have already been published in Wang et al. (1999), many Lévy plate results of different boundary and loading conditions have been omitted due to the length limitation of a technical brief note. Furthermore, there is no further explanation on the errors made by Cooke and Levinson (1983) in Wang et al. (1999), except with some numerical illustration. Hence this study wish to furnish these unpublished plate results together with a more general derivation for the Kirchhoff–Mindlin bending relationships. In addition, the errors by Cooke and Levinson (1983) will be clarified in detail. To evaluate the correctness of the bending relationships, the numerical plate results by the present work are validated with those obtained by other researchers mentioned earlier and a well known commercial software, ABAQUS (1997).

Finally, the shear deformable plate theory proposed by Reissner (1944, 1945, 1947) will be contrasted against the Mindlin plate theory. It is generally perceived by some researchers that when the Mindlin shear force correction factor is set to 5/6, the Mindlin plate theory reduces to the Reissner plate theory. On the contrary to this perception, it will be shown herein that these two plate theories are dissimilar in their formulations and assumptions which can significantly change the results. The dissimilarities between the two plate theories will be substantiated with the numerical results computed in the present work and those from Salerno and Goldberg (1960) using the Reissner plate theory.

2. Mindlin plate theory

Based on the Mindlin plate theory, the stress resultants of a plate in bending satisfy the following equations of equilibrium (Timoshenko and Woinowsky-Krieger, 1959)

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \tag{1a}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \tag{1b}$$

$$\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0. \tag{1c}$$

From the constitutive equations, the Mindlin stress resultants in terms of the displacement and rotations may be written as (Reismann, 1988)

$$M_{xx}^{\mathrm{M}} = D\left(\frac{\partial \phi_x}{\partial x} + v\frac{\partial \phi_y}{\partial y}\right), \tag{2a}$$

$$M_{yy}^{\mathrm{M}} = D\left(v\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right), \tag{2b}$$

$$M_{xy}^{\mathrm{M}} = \frac{1}{2}D(1-v)\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right), \tag{2c}$$

$$Q_x^{\mathrm{M}} = \kappa^2 Gh\left(\phi_x + \frac{\partial w^{\mathrm{M}}}{\partial x}\right), \tag{2d}$$

$$Q_y^{\mathrm{M}} = \kappa^2 Gh\left(\phi_y + \frac{\partial w^{\mathrm{M}}}{\partial y}\right), \tag{2e}$$

where ϕ_x and ϕ_y denote the rotations of the normals to the middle surface about the y- and x-axes, respectively, and w^{M} is the transverse deflection of the middle surface of the plate. The superscript M denotes Mindlin plate quantities. Also, D, G and v are the plate bending rigidity, shear modulus and Poisson's ratio, respectively. In Eqs. (2d) and (2e), κ^2 is shear correction factor that has been introduced to modify the shear modulus. This shear correction factor depends not only on the material and geometric parameters but also on the loading and boundary conditions. Generally for rectangular plates of uniform thickness, κ^2 can be taken to be 5/6 and this value has been adopted throughout this study.

As the Mindlin plate theory includes the effect of transverse shear deformation, the Mindlin shear forces can be obtained from the constitutive relationships, as shown in Eqs. (2d) and (2e). One may also determine the Mindlin shear forces by substituting the bending and twisting moments in Eqs. (2a)–(2c) into the equilibrium equations, Eqs. (1b) and (1c), respectively. With that, the Mindlin equilibrium shear forces are

$$Q_x^{\mathrm{M}} = D\left[\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + \frac{1}{2}(1-v)\frac{\partial}{\partial y}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right)\right], \tag{3a}$$

$$Q_y^{\mathrm{M}} = D\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) - \frac{1}{2}(1-v)\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right)\right]. \tag{3b}$$

Introducing the *Marcus* moment or moment sum $\mathsf{M} = (M_{xx} + M_{yy})/(1+v)$, Eqs. (3a) and (3b) become

$$Q_x^{\mathrm{M}} = \frac{\partial \mathsf{M}^{\mathrm{M}}}{\partial x} + \frac{1}{2}D(1-v)\frac{\partial}{\partial y}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right), \tag{4a}$$

$$Q_y^{\mathrm{M}} = \frac{\partial \mathsf{M}^{\mathrm{M}}}{\partial y} - \frac{1}{2}D(1-v)\frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right). \tag{4b}$$

In view of Eqs. (1a), (4a) and (4b), the governing equation of a Mindlin plate under bending can be written as

$$\nabla^2 \mathsf{M}^{\mathrm{M}} = -q \Rightarrow \nabla^2\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) = -\frac{q}{D}, \tag{5}$$

where $\nabla^2(\cdot) = \partial^2(\cdot)/\partial x^2 + \partial^2(\cdot)/\partial y^2$ is the Laplacian operator.

The Mindlin governing equation may also be obtained by considering the transverse shear forces in Eqs. (2d) and (2e), and the equilibrium equation in Eq. (1a). This gives

$$\kappa^2 Gh\left(\nabla^2 w^{\mathrm{M}} + \frac{\mathsf{M}^{\mathrm{M}}}{D}\right) = -q. \tag{6}$$

It can also be noted that equating the shear forces in Eqs. (2d) and (2e) to those in Eqs. (4a) and (4b), respectively, and eliminating the Marcus moment in the process, one can deduce the following equation:

$$\nabla^2\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right) = c^2\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right), \tag{7}$$

where $c^2 = 2\kappa^2 Gh/[D(1-v)]$.

Finally, the uncoupled equation for the transverse deflection can be obtained by combining Eqs. (5) and (6) to give

$$D\nabla^4 w^M = \left(1 - \frac{D}{\kappa^2 Gh}\nabla^2\right) q. \tag{8}$$

At the same time, in view of Eqs. (5) and (7), the expressions for the normal rotations may be written as

$$\nabla^4 \phi_x = -\frac{1}{D}\frac{\partial q}{\partial x} + c^2 \frac{\partial}{\partial y}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right), \tag{9a}$$

$$\nabla^4 \phi_y = -\frac{1}{D}\frac{\partial q}{\partial y} - c^2 \frac{\partial}{\partial x}\left(\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x}\right). \tag{9b}$$

Eq. (8) is the well-known uncoupled equation for the Mindlin deflection, and is the same as that derived by Cooke and Levinson (1983). However, the uncoupled equations for the normal rotations by Cooke and Levinson (1983) do not contain the last term on the right hand side of Eqs. (9a) and (9b). In the open literature, the ignorance of the last term was sometimes taken as a common approximation. Rigorously, this last term can only be omitted for the special case of a polygonal plate with all edges simply supported. This is because the missing terms in Cooke and Levinson vanish for such plates, as will be substantiated later in this paper. As reported by Reddy et al. (2001) for Lévy plates, to ignore the last term will have an undesirable stiffening effect on the plate bending behaviour. It should also be emphasised that Eqs. (5)–(9b) are not alternative governing equations to the three given in Eqs. (1a)–(1c) and are presented above for subsequent development of the bending relationships.

3. Kirchhoff plate theory

For Kirchhoff plates, the equilibrium equations are similar to Eqs. (1a)–(1c), and the Kirchhoff stress resultants in terms of the displacement are (Reismann, 1988)

$$M_{xx}^{\mathrm{K}} = -D\left(\frac{\partial^2 w^{\mathrm{K}}}{\partial x^2} + \nu\frac{\partial^2 w^{\mathrm{K}}}{\partial y^2}\right), \tag{10a}$$

$$M_{yy}^{\mathrm{K}} = -D\left(\nu\frac{\partial^2 w^{\mathrm{K}}}{\partial x^2} + \frac{\partial^2 w^{\mathrm{K}}}{\partial y^2}\right), \tag{10b}$$

$$M_{xy}^{\mathrm{K}} = -D(1-\nu)\frac{\partial^2 w^{\mathrm{K}}}{\partial x\,\partial y}. \tag{10c}$$

where the superscript K denotes Kirchhoff plate quantities. Since the classical thin plate theory neglects the presence of transverse shear strain, the transverse shear forces can only be determined from the equilibrium equations. In view of Eqs. (1b), (1c) and (10a)–(10c), these shear forces can be expressed in term of the Marcus moment as

$$Q_x^{\mathrm{K}} = -D\frac{\partial}{\partial x}(\nabla^2 w^{\mathrm{K}}) = \frac{\partial \mathsf{M}^{\mathrm{K}}}{\partial x}, \tag{11a}$$

$$Q_y^{\mathrm{K}} = -D\frac{\partial}{\partial y}(\nabla^2 w^{\mathrm{K}}) = \frac{\partial \mathsf{M}^{\mathrm{K}}}{\partial y}. \tag{11b}$$

From Eqs. (1a), (11a) and (11b), the well-known fourth-order governing equation of the Kirchhoff plate theory can be established as

$$\nabla^4 w^{\mathrm{K}} = \frac{q}{D} \Rightarrow \nabla^2 \mathsf{M}^{\mathrm{K}} = -q. \tag{12}$$

4. General Kirchhoff–Mindlin bending relationships

It is observed that regardless of the plate theories, the transverse load acting on the plate is the same. With this concept of load equivalence, Eqs. (5) and (12) lead to the Kirchhoff–Mindlin Marcus moment relationship:

$$\nabla^2 \mathsf{M}^{\mathrm{M}} = \nabla^2 \mathsf{M}^{\mathrm{K}} \Rightarrow \mathsf{M}^{\mathrm{M}} = \mathsf{M}^{\mathrm{K}} + D\nabla^2\Phi, \tag{13}$$

where $\Phi(x,y)$ is a bi-harmonic function that satisfies $\nabla^4\Phi = 0$.

In view of Eqs. (6), (12) and (13) and solving the final expression, one can obtain the Kirchhoff–Mindlin deflection relationship as

$$w^{\mathrm{M}} = w^{\mathrm{K}} + \frac{\mathsf{M}^{\mathrm{K}}}{\kappa^2 Gh} + \Psi - \Phi, \tag{14}$$

where $\Psi(x,y)$ is a harmonic function that satisfies the Laplace equation $\nabla^2\Psi = 0$. In order to link the Mindlin normal rotations to the Kirchhoff slopes, the differential equation, given in Eq. (7) is first solved and its solution can be generally expressed as $\Omega(x,y)$. Then by substituting Eqs. (2d), (13) and (14) into Eq. (4a), it can be found that

$$\phi_x = -\frac{\partial w^K}{\partial x} + \frac{\partial}{\partial x}\left[\frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi\right] + \frac{1}{c^2}\frac{\partial \Omega}{\partial y}, \tag{15a}$$

and similarly from Eqs. (2e), (4b), (13) and (14),

$$\phi_y = -\frac{\partial w^K}{\partial y} + \frac{\partial}{\partial y}\left[\frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi\right] - \frac{1}{c^2}\frac{\partial \Omega}{\partial x}. \tag{15b}$$

It is to note that one can also express the Mindlin rotations as a vector field via the Helmholtz representation as presented in Appendix A. While such a representation yields simpler expressions, one need to solve thick plate governing equations to determine the associated potentials.

Now, to furnish the Kirchhoff–Mindlin stress-resultant relationships, one can simply substitute Eqs. (14)–(15b) into Eqs. (2a)–(2e) and obtain

$$M_{xx}^M = M_{xx}^K - D(1-\nu)\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{D}{\kappa^2 Gh}\nabla^2\Phi + \Phi - \Psi\right) - \frac{1}{c^2}\frac{\partial \Omega}{\partial x}\right] + D\nabla^2\Phi, \tag{16a}$$

$$M_{yy}^M = M_{yy}^K - D(1-\nu)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{D}{\kappa^2 Gh}\nabla^2\Phi + \Phi - \Psi\right) + \frac{1}{c^2}\frac{\partial \Omega}{\partial y}\right] + D\nabla^2\Phi, \tag{16b}$$

$$M_{xy}^M = M_{xy}^K + D(1-\nu)\left[\frac{\partial^2}{\partial x \partial y}\left(\frac{D}{\kappa^2 Gh}\nabla^2\Phi + \Phi - \Psi\right) + \left(\frac{1}{2} - \frac{1}{c^2}\frac{\partial^2}{\partial x^2}\right)\Omega\right], \tag{16c}$$

$$Q_x^M = Q_x^K + \frac{\partial}{\partial x}(D\nabla^2\Phi) + \frac{D(1-\nu)}{2}\frac{\partial \Omega}{\partial y}, \tag{16d}$$

$$Q_y^M = Q_y^K + \frac{\partial}{\partial y}(D\nabla^2\Phi) - \frac{D(1-\nu)}{2}\frac{\partial \Omega}{\partial x}. \tag{16e}$$

With that, the exact Kirchhoff–Mindlin bending relationships have been derived. They contain intrinsic plate functions Φ, Ψ and Ω that may be established uniquely for any plate shape of general boundary and loading conditions. In the next section, these exact Kirchhoff–Mindlin bending relationships will be specialised for rectangular Lévy plates.

5. Lévy plate formulation

Consider an isotropic and homogeneous rectangular Lévy plate with uniform thickness h, length a, width b, modulus of elasticity E, Poisson's ratio ν, and shear modulus $G = E/[2(1+\nu)]$. Adopting the rectangular Cartesian coordinate system as shown in Fig. 1 with its origin at the mid-left edge of the plate, the edges $x = 0$ and $x = a$ are simply supported while the other two edges along $y = \pm b/2$ can be of any combination of simply supported, clamped or free conditions. The form of the transverse loading on the Lévy plate may be characterised by

$$q(x,y) = \sum_{m=1}^{\infty} q_m(y) \sin\frac{m\pi x}{a}. \tag{17}$$

The boundary conditions along the edges $x = 0$ and $x = a$ and the transverse loading in Eq. (17) can be satisfied by the displacement functions of the form:

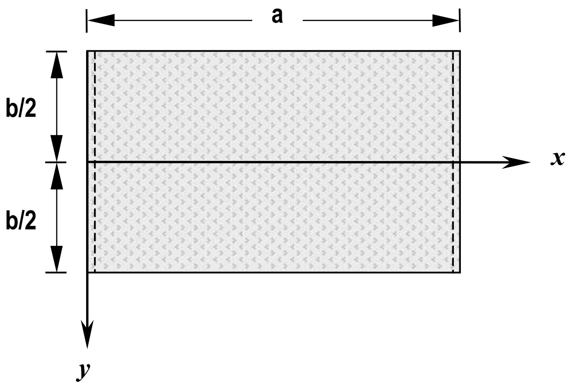

Fig. 1. Lévy plate and the rectangular coordinate system.

$$w^{\mathrm{M}}(x,y) = \sum_{m=1}^{\infty} \overline{w}_m^{\mathrm{M}}(y) \sin\frac{m\pi x}{a}, \tag{18a}$$

$$\phi_x(x,y) = \sum_{m=1}^{\infty} \varphi_{xm}(y) \cos\frac{m\pi x}{a}, \tag{18b}$$

$$\phi_y(x,y) = \sum_{m=1}^{\infty} \varphi_{ym}(y) \sin\frac{m\pi x}{a}, \tag{18c}$$

whereas the Kirchhoff displacement function can be taken as

$$w^{\mathrm{K}}(x,y) = \sum_{m=1}^{\infty} \overline{w}_m^{\mathrm{K}}(y) \sin\frac{m\pi x}{a}. \tag{19}$$

In view of Eqs. (17)–(19), the intrinsic plate functions for Lévy plates are

$$\Phi(x,y) = \sum_{m=1}^{\infty} \frac{ay}{2m\pi}\left(C_{1m}\cosh\frac{m\pi y}{a} + C_{2m}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}, \tag{20a}$$

$$\Psi(x,y) = \sum_{m=1}^{\infty} \left(C_{3m}\cosh\frac{m\pi y}{a} + C_{4m}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a}, \tag{20b}$$

$$\Omega(x,y) = \sum_{m=1}^{\infty} (C_{5m}\sinh\lambda_m y + C_{6m}\cosh\lambda_m y)\cos\frac{m\pi x}{a}, \tag{20c}$$

where $\lambda_m^2 = (m\pi/a)^2 + c^2$ and C_{im} $(i = 1, \ldots, 6)$ are constants of integration that can be determined by applying the boundary conditions along the edges $y = \pm b/2$. Similar bending relationships can also be found in Wang et al. (1999). Based on the Mindlin plate theory, there are three boundary conditions that are to be satisfied along each of the two edges. These six independent boundary conditions are sufficient for determining all the six unknown constants. Below, these constants have been evaluated for Lévy plates with simply supported edges along $y = \pm b/2$. One can refer to Appendix B for the constants of Lévy plates with other combinations of edge conditions along $y = \pm b/2$.

5.1. SSSS Lévy plates

When the other two plate edges of the Lévy plate are also simply supported, the plate reduces to a Navier plate with the following boundary conditions at the edges $y = \pm b/2$:

$$w^{M} = w^{K} = 0, \qquad M_{yy}^{M} = M_{yy}^{K} = 0, \qquad \phi_x = 0. \tag{21a–c}$$

In view of Eqs. (14), (15a), (16b) and (20a)–(20c), it can be shown that

$$C_{1m} = C_{2m} = C_{3m} = C_{4m} = C_{5m} = C_{6m} = 0. \tag{22}$$

Therefore, for simply supported Lévy plates (or Navier plates), the bending relationships are given by

$$w^{M} = w^{K} + \frac{\mathsf{M}^{K}}{\kappa^2 Gh}, \tag{23a}$$

$$\phi_x = -\frac{\partial w^{K}}{\partial x}, \qquad \phi_y = -\frac{\partial w^{K}}{\partial y}, \tag{23b, c}$$

$$M_{xx}^{M} = M_{xx}^{K}, \qquad M_{yy}^{M} = M_{yy}^{K}, \tag{23d, e}$$

$$M_{xy}^{M} = M_{xy}^{K}, \qquad Q_x^{M} = Q_x^{K}, \qquad Q_y^{M} = Q_y^{K}. \tag{23f–h}$$

It can be shown that the above deflection relationship, Eq. (23a) is also valid for even other than Lévy-type loading conditions (Wang and Alwis, 1995).

Due to Eq. (22) where $C_{5m} = C_{6m} = 0$, it can be deduced that the intrinsic plate function Ω in Eq. (20c) also becomes zero, and hence Eq. (7) can be reduced to

$$\frac{\partial \phi_x}{\partial y} = \frac{\partial \phi_y}{\partial x}. \tag{24}$$

This therefore validates the reasoning discussed earlier concerning Eqs. (9a) and (9b) that the uncoupled equations for the normal rotations by Cooke and Levinson (1983) are valid for Navier plates only.

6. Mindlin and Reissner plate theories

To stipulate the kinematic behaviour through the plate thickness that includes the effect of the transverse shear deformation, both Mindlin (1951) and Reissner (1944, 1945, 1947) had independently proposed a shear deformable plate theory for analysing moderately thick plates. Many researchers still have the perception that both theories are essentially similar especially when the Mindlin shear correction factor is taken to be 5/6 and often associate them together as the Reissner–Mindlin plate theory. To an extent, some researchers may have adopted one of these plate theories in their plate analyses and yet compare their numerical solutions against those furnished using the other plate theory.

As such, it is useful to establish the dissimilarity between both plate theories in terms of the Kirchhoff quantities. A major distinction between both theories is that the Reissner plate theory is derived from the variational principle of complementary strain energy with the assumption of linear bending stress distribution and parabolic transverse shear stress distribution through the thickness. On the other hand, Mindlin formulated his theory by first assuming a linear variation of displacement across the plate thickness while maintaining the transverse inextensibility of the plate thickness. The formulation of the Reissner plate theory leads to the displacement variation not necessarily being linear across the plate thickness and also the plate thickness being deformed. It is only through the matching of the actual work done by the stresses to that by equivalent moments or forces that Reissner obtained the weighted average or equivalent values of displacements across the plate thickness. On reviewing the formulation of both plate theories, it is again evident that the normal stress σ_{zz} has been ignored in the Mindlin plate theory as opposed to its existence in the Reissner plate theory.

As these differences between both plate theories have been corroborated earlier (Panc, 1975), the purpose of this section is to treat this problem in a unique way, that is, by first seeking the bending relationships between the Mindlin and Reissner plate quantities and then between Kirchhoff and Reissner plate quantities. Any difference between the plate theories can be seen easily from the bending relationships.

Now, the Reissner stress resultants are given as follows (Reissner, 1947):

$$M_{xx}^{\mathrm{R}} = -D\left(\frac{\partial^2 w^{\mathrm{R}}}{\partial x^2} + v\frac{\partial^2 w^{\mathrm{R}}}{\partial y^2}\right) + \frac{D(1-v)}{C_s}\frac{\partial Q_x^{\mathrm{R}}}{\partial x} + D\left(\frac{1+v}{C_n} - \frac{v}{C_s}\right)q, \tag{25a}$$

$$M_{yy}^{\mathrm{R}} = -D\left(v\frac{\partial^2 w^{\mathrm{R}}}{\partial x^2} + \frac{\partial^2 w^{\mathrm{R}}}{\partial y^2}\right) + \frac{D(1-v)}{C_s}\frac{\partial Q_y^{\mathrm{R}}}{\partial y} + D\left(\frac{1+v}{C_n} - \frac{v}{C_s}\right)q, \tag{25b}$$

$$M_{xy}^{\mathrm{R}} = -D(1-v)\frac{\partial^2 w^{\mathrm{R}}}{\partial x\,\partial y} + \frac{D(1-v)}{2C_s}\left(\frac{\partial Q_x^{\mathrm{R}}}{\partial y} + \frac{\partial Q_y^{\mathrm{R}}}{\partial x}\right), \tag{25c}$$

$$\left[1 - \frac{D(1-v)}{2C_s}\nabla^2\right]Q_x^{\mathrm{R}} = -D\frac{\partial}{\partial x}(\nabla^2 w^{\mathrm{R}}) - D(1+v)\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial q}{\partial x}, \tag{25d}$$

$$\left[1 - \frac{D(1-v)}{2C_s}\nabla^2\right]Q_y^{\mathrm{R}} = -D\frac{\partial}{\partial y}(\nabla^2 w^{\mathrm{R}}) - D(1+v)\left(\frac{1}{2C_s} - \frac{1}{C_n}\right)\frac{\partial q}{\partial y}, \tag{25e}$$

where the superscript R denotes the Reissner plate quantities. For homogeneous plates, C_s and C_n can be found to be $5Gh/6$ and $5Eh/6v$, respectively (Reissner, 1947). Since the equilibrium equations (1a)–(1c) also apply to the Reissner plate theory, Eqs. (1a), (25d) and (25e) can be combined to provide the fourth-order governing equation for the Reissner plate deflection

$$D\nabla^4 w^{\mathrm{R}} = q - D\left(\frac{1}{C_s} - \frac{1+v}{C_n}\right)\nabla^2 q. \tag{26}$$

From Eqs. (25a) and (25b), the Marcus moment in the Reissner plate theory is given by

$$\mathsf{M}^{\mathrm{R}} = -D\nabla^2 w^{\mathrm{R}} - D\left(\frac{1}{C_s} - \frac{2}{C_n}\right)q. \tag{27}$$

Subsequently, by substituting Eq. (27) into Eq. (26), the Reissner governing equation in terms of the Marcus moment can be expressed as follows:

$$\nabla^2\left[\mathsf{M}^{\mathrm{R}} - \frac{D(1-v)}{C_n}q\right] = -q. \tag{28}$$

Using Eqs. (5) and (28) and based on the concept of load equivalence, the Marcus moment relationship between the Mindlin and Reissner plate theories is given by

$$\mathsf{M}^{\mathrm{R}} = \mathsf{M}^{\mathrm{M}} + \frac{D(1-v)}{C_n}q + D\nabla^2\Theta, \tag{29}$$

where $\Theta(x, y)$ has the bi-harmonic property that satisfies $\nabla^4\Theta = 0$. Then, with the substitution of Eq. (29) into Eq. (27), and together with Eqs. (5) and (6), the Mindlin–Reissner deflection relationship can be furnished as

$$w^{\mathrm{R}} = w^{\mathrm{M}} + \left[\frac{1}{C_s} - \frac{1+v}{C_n} - \frac{1}{\kappa^2 Gh}\right]\mathsf{M}^{\mathrm{M}} - \Theta + \Lambda, \tag{30}$$

where $\nabla^2\Lambda = 0$. By considering the special case of a Navier plate where the Marcus moment vanishes along the straight simply supported plate edges, the intrinsic plate functions Θ and Λ will disappear. Thus, for a homogeneous Navier plate, by taking note of the definition of C_s in the Reissner plate theory and by setting $\kappa^2 = 5/6$ in the Mindlin plate theory, the Mindlin–Reissner deflection relationship reduces

$$w^{\mathrm{R}} = w^{\mathrm{M}} - \frac{1+v}{C_n}\mathsf{M}^{\mathrm{M}}. \tag{31}$$

By considering Eqs. (23a), (23d,e) and (31), one can also deduce the Kirchhoff–Reissner deflection relationship as

$$w^{\mathrm{R}} = w^{\mathrm{K}} + \left[1 - (1+v)\frac{\kappa^2 Gh}{C_n}\right]\frac{\mathsf{M}^{\mathrm{K}}}{\kappa^2 Gh}. \tag{32}$$

Hence from Eqs. (23a), (31) and (32) and by taking note of the definition of C_n, it can be seen evidently that there is considerable difference in the prediction of deflection by both shear deformable plate theories as long as the strain energy due to ε_{zz} and σ_{zz} is not ignored.

For Lévy plates with other boundary conditions along $y = \pm b/2$, it is comprehensible that the intrinsic plate functions Θ and Λ do not vanish. To elucidate the distinction in the plate solutions furnished by both plate theories, numerical results on the transverse deflection by Salerno and Goldberg (1960) have been tabulated in Table 1 against the present solutions from the Kirchhoff–Mindlin deflection relationship in Eqs. (14), (20a)–(20c) and (B.5a)–(B.5f), and those generated by ABAQUS (1997) using Mindlin shell (*S8R*) elements with $\kappa^2 = 5/6$. Table 1 presents the non-dimensionalised deflection $[\overline{w} = wD/(q_o a^4)]$ for square SFSF Lévy plates for different values of h/a, under uniformly distributed load, q_o, with v taken as 0.3.

From Table 1, it can be seen clearly that the plate deflections predicted by Salerno and Goldberg (1960) using the Reissner plate theory are consistently lower than the present Mindlin plate results. The results presented in Table 1 and all the above-mentioned justifications show clearly the differences between the two plate theories. Hence, it is important to differentiate both shear deformable plate theories appropriately when applied to thick plate analysis.

Table 1
Non-dimensionalised deflection of SFSF Lévy plates

h/a	At the centre of the plate, $\overline{w}$ $(a/2, 0)$			At the mid-span of the free edge, $\overline{w}$ $(a/2, a/2)$		
	Salerno and Goldberg (1960)	ABAQUS (1997)[a]	Present results	Salerno and Goldberg (1960)	ABAQUS (1997)[a]	Present results
0.10	0.01341	0.01346	**0.01346**	0.01557	0.01560	**0.01560**
0.15	0.01379	0.01391	**0.01391**	0.01609	0.01616	**0.01616**
0.20	0.01433	0.01454	**0.01454**	0.01678	0.01690	**0.01690**
0.25	0.01502	0.01535	**0.01536**	0.01762	0.01781	**0.01781**
0.30	0.01586	0.01633	**0.01633**	0.01864	0.01889	**0.01889**

[a] Using 40 × 40 *S8R* shell elements based on the Mindlin plate theory.

7. Numerical examples and verification

With the establishment of the Kirchhoff–Mindlin bending relationships in Eqs. (14)–(17) and with the constants of integration uniquely defined for Lévy plates of various boundary conditions in Eqs. (22)–(24) and (B.1a–c)–(B.10f), numerical solutions can be generated. Herein, Mindlin plate results will be furnished for square Lévy plates with symmetrical boundary conditions, i.e., SSSS, SCSC and SFSF types. Wherever possible, results from past research are used for comparison (Kant and Hinton, 1980; Kant, 1982; Yuan and Miller, 1992; Dong et al., 1993; Dong and Teixeira de Freitas, 1994). Plate solutions computed with ABAQUS (1997) using Mindlin shell (*S8R*) elements are also used for validation. For uniformly loaded plates, harmonic number m is chosen to ensure convergence and a valid comparison. Also, v is taken to be 0.3 while the shear correction factor κ^2 is 5/6.

The non-dimensionalised central deflection $[\overline{w} = wD/(q_o a^4)]$ of uniformly loaded square Lévy plates with symmetrical boundary conditions ($h/a = 0.2$) have been tabulated and presented in Table 2. From Table 2, it is observed that the present results show very good agreement with the results furnished by other researchers, except those from Kant (1982). This is because Kant (1982) used a higher-order plate theory that allows for a quadratic variation of transverse shear strain and a linear distribution of the transverse normal strain.

In the comparison of the non-dimensionalised deflections with respect to different plate thicknesses, plate solutions of SFSF Lévy plates, obtained by Dong et al. (1993) and Dong and Teixeira de Freitas (1994) are presented against the present results in Table 3. Again, a good trend of agreement can be observed with results from Dong et al. (1993) being comparatively closer to the present solutions.

Another way of checking the correctness of the present Kirchhoff–Mindlin bending relationships is to compare the stress resultants. Thus, numerical plate solutions for the stress resultants are furnished using the above relationships together with those determined by Kant and Hinton (1980) for different plate thicknesses. Tables 4–6 show the tabulated results for the Mindlin stress resultants of SSSS-, SCSC- and

Table 2
Non-dimensionalised central deflection of square symmetrical Lévy plates ($h/a = 0.2, m = 40$)

	SSSS	SCSC	SFSF
Kant (1982)	0.004800	0.002930	0.014304
Kant and Hinton (1980)	0.004900	0.003016	0.014496
Yuan and Miller (1992)	0.004905	0.003021	0.014542
ABAQUS (1997)[a]	0.004905	0.003021	0.014539
Present results	**0.004904**	**0.003021**	**0.014539**

[a] Using 40 × 40 *S8R* shell elements based on the Mindlin plate theory.

Table 3
Non-dimensionalised deflection of square SFSF Lévy plates with different thicknesses ($m = 40$)

h/a	At centre of plate			At mid-span of free edge		
	Dong et al. (1993)	Dong and Teixeira de Freitas (1994)	Present results	Dong et al. (1993)	Dong and Teixeira de Freitas (1994)	Present results
0.10	0.01346	0.01340	**0.01346**	0.01562	0.01549	**0.01560**
0.15	0.01391	0.01385	**0.01391**	0.01617	0.01607	**0.01616**
0.20	0.01454	0.01448	**0.01454**	0.01690	0.01679	**0.01690**
0.25	0.01535	0.01528	**0.01535**	0.01781	0.01771	**0.01781**
0.30	0.01633	0.01627	**0.01633**	0.01890	0.01879	**0.01889**

Table 4
Non-dimensionalised stress resultants of square SSSS Lévy plates with different thicknesses

Point $(x/a, y/a)$	Stress resultants	Kirchhoff plate solution	$h/a = 0.01$–0.2	
			Kant and Hinton (1980)	Present results
(0.5, 0)	$M_{xx}/(q_o a^2)$	0.0479	0.0479	**0.0479**
(0.5, 0)	$M_{yy}/(q_o a^2)$	0.0479	0.0478	**0.0479**
(1.0, 0.5)	$M_{xy}/(q_o a^2)$	0.0325	0.0324	**0.0325**
(1.0, 0)	$Q_x/(q_o a)$	0.333	0.332	**0.333**
(0.5, 0.5)	$Q_y/(q_o a)$	0.338	0.337	**0.338**

Table 5
Non-dimensionalised stress resultants of square SCSC Lévy plates with different thicknesses

Point $(x/a, y/a)$	Stress resultants	Kirchhoff plate solution	$h/a = 0.02$		$h/a = 0.1$		$h/a = 0.2$	
			Kant and Hinton (1980)	Present results	Kant and Hinton (1980)	Present results	Kant and Hinton (1980)	Present results
(0.5, 0)	$M_{xx}/(q_o a^2)$	0.0244	0.0244	**0.0244**	0.0258	**0.0258**	0.0292	**0.0292**
(0.5, 0)	$M_{yy}/(q_o a^2)$	0.0332	0.0332	**0.0332**	0.0332	**0.0333**	0.0330	**0.0331**
(0.5, 0.5)	$M_{yy}/(q_o a^2)$	0.0698	0.0697	**0.0698**	0.0679	**0.0680**	0.0626	**0.0627**
(1.0, 0)	$Q_x/(q_o a)$	0.239	0.239	**0.240**	0.243	**0.243**	0.251	**0.251**
(0.5, 0.5)	$Q_y/(q_o a)$	0.516	0.513	**0.513**	0.500	**0.500**	0.475	**0.475**

Table 6
Non-dimensionalised stress resultants of square SFSF Lévy plates with different thicknesses

Point $(x/a, y/a)$	Stress resultants	Kirchhoff plate solution	$h/a = 0.02$		$h/a = 0.1$		$h/a = 0.2$	
			Kant and Hinton (1980)	Present results	Kant and Hinton (1980)	Present results	Kant and Hinton (1980)	Present results
(0.5, 0)	$M_{xx}/(q_o a^2)$	0.123	0.122	**0.123**	0.122	**0.122**	0.123	**0.123**
(0.5, 0)	$M_{yy}/(q_o a^2)$	0.0271	0.0268	**0.0268**	0.0256	**0.0256**	0.0237	**0.0237**
(1.0, 0)	$Q_x/(q_o a)$	0.464	0.463	**0.463**	0.460	**0.460**	0.456	**0.457**

SFSF-Lévy plates of different plate thicknesses, respectively. The number of harmonics used for generating the present results using the bending relationships is 40.

8. Concluding remarks

Presented herein are the exact Kirchhoff–Mindlin bending relationships developed with the concept of load equivalence. These bending relationships contain general intrinsic plate functions that can be applied to any plate shape of arbitrary boundary and loading conditions. Once specialised, these bending relationships can be used to furnish exact Mindlin plate results upon supplying the widely available corresponding Kirchhoff plate solutions. In this study, these Kirchhoff–Mindlin bending relationships have been specialised for Lévy plates with different boundary conditions. The numerical plate results generated using these relationships show very good correspondence with other solutions from past research. From the derivation, it was discovered that the analytical Mindlin results for Lévy plates by Cooke and Levinson

(1983) are erroneous. Their errors have been established in this study and are traced to missing terms in the uncoupled governing equations for the normal rotations.

Also highlighted in this study are the differences between the Reissner and Mindlin plate theories in terms of the corresponding Kirchhoff quantities. The most important difference is that the Reissner plate theory allows for transverse extensibility while the Mindlin plate theory does not. This difference can be quantified explicitly for Navier plates.

Appendix A

One can always express ϕ_x and ϕ_y as a vector field $\boldsymbol{\phi} = [\,\phi_x \quad \phi_y \quad 0\,]^{\mathrm{T}}$. Since $\boldsymbol{\phi}$ must be continuously differentiable, it can be represented as the sum of an irrotational vector field which is the gradient of a scalar potential A and a solenodial vector field which is the curl of the vector potential $\mathbf{B} = [\,B_1 \quad B_2 \quad B_3\,]^{\mathrm{T}}$ (Malvern, 1969)

$$\boldsymbol{\phi} = \vec{\nabla} A + \vec{\nabla} \times \mathbf{B}, \tag{A.1}$$

where $\vec{\nabla}$ is the del differential operator. The scalar potential A and vector potential $\mathbf{B}$ must satisfy the following conditions:

$$\vec{\nabla} \times (\vec{\nabla} A) = \mathbf{0}, \tag{A.2a}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{B}) = 0, \tag{A.2b}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0. \tag{A.2c}$$

The way of expressing $\boldsymbol{\phi}$ as shown in Eq. (A.1) is commonly known as the Helmholtz representation. Note that Eq. (A.2c) is an additional requirement imposed on $\mathbf{B}$ to facilitate subsequent simplification; such an imposition limits the choice of $\mathbf{B}$. For the problem concerned, one can further assume that the potentials A and $\mathbf{B}$ to be independent of z and this leads to the k-component of the column vector $(\vec{\nabla} \times \mathbf{B})$ in Eq. (A.1) to be zero.

Taking the curl of Eq. (A.1) and from Eqs. (A.2a) and (A.2c), one obtains

$$\frac{\partial \phi_x}{\partial y} - \frac{\partial \phi_y}{\partial x} = \nabla^2 B_3, \tag{A.3}$$

while the divergence of Eq. (A.1) gives the following:

$$\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} = \nabla^2 A. \tag{A.4}$$

Herein to simplify subsequent algebraic manipulation, one may want to put the potential A in terms of the Kirchhoff transverse deflection w^{K} as

$$A = -w^{\mathrm{K}} + \tilde{A}. \tag{A.5}$$

Substituting Eq. (A.6) into Eq. (5) reveals the bi-harmonic nature of $\tilde{A}$

$$\nabla^4 \tilde{A} = 0. \tag{A.6}$$

Now from Eqs. (A.1) and (A.5), the Mindlin rotations can be re-written as

$$\phi_x(x, y) = -\frac{\partial w^{\mathrm{K}}}{\partial x} + \frac{\partial \tilde{A}}{\partial x} + \frac{\partial B_3}{\partial y}, \tag{A.7a}$$

$$\phi_y(x,y) = -\frac{\partial w^{\mathrm{K}}}{\partial y} + \frac{\partial \tilde{A}}{\partial y} - \frac{\partial B_3}{\partial x}. \tag{A.7b}$$

To determine $\tilde{A}$ and B_3, one can compare Eqs. (A.7a) and (A.7b) to the rotation–slope relationships given in Eqs. (15a) and (15b) and obtain

$$\tilde{A} = \frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi, \tag{A.8a}$$

$$B_3 = \left(\frac{1}{c^2}\right)\Omega. \tag{A.8b}$$

It is clear that the expression for $\tilde{A}$ given in Eq. (A.8a) satisfies the condition as stated in Eq. (A.6). While the value of B_1 and B_2 play no role in the above derivation, they can be taken to be zero to ensure the adherence of Eq. (A.2c), without any loss of generality. With $\tilde{A}$ and B_3 now being established, one can proceed to derive the bending relationships as presented in Eqs. (13)–(16e).

Appendix B

The constants of integration for plates with edges of various boundary conditions are given as below. Note that these constants are valid for the types of loadings whose shape is to be the same for all sections that are parallel to the two simply supported edges.

B.1. SCSC Lévy plates

For the clamped edges at $y = \pm b/2$, the boundary conditions are

$$w^{\mathrm{M}} = w^{\mathrm{K}} = 0, \qquad \phi_y = \frac{\partial w^{\mathrm{K}}}{\partial y} = 0, \qquad \phi_x = 0. \tag{B.1a–c}$$

The substitution of the boundary conditions Eqs. (B.1a–c) into Eqs. (14)–(15b) and (20a)–(20c) gives

$$C_{1m} = \frac{\mu_m^-\left(\coth\frac{m\pi b}{2a}\sinh\frac{\lambda_m b}{2} - \frac{m\pi}{a\lambda_m}\cosh\frac{\lambda_m b}{2}\right)}{\left\{\frac{m\pi}{a\lambda_m}\frac{D}{\kappa^2 Gh}\sinh\frac{m\pi b}{2a}\cosh\frac{\lambda_m b}{2} - \left[\frac{D}{\kappa^2 Gh} + \frac{1}{2}\left(\frac{a}{m\pi}\right)^2\right]\cosh\frac{m\pi b}{2a}\sinh\frac{\lambda_m b}{2} + \frac{ab}{4m\pi}\operatorname{csch}\frac{m\pi b}{2a}\sinh\frac{\lambda_m b}{2}\right\}}, \tag{B.2a}$$

$$C_{2m} = \frac{\mu_m^+\left(\tanh\frac{m\pi b}{2a}\cosh\frac{\lambda_m b}{2} - \frac{m\pi}{a\lambda_m}\sinh\frac{\lambda_m b}{2}\right)}{\left\{\frac{m\pi}{a\lambda_m}\frac{D}{\kappa^2 Gh}\cosh\frac{m\pi b}{2a}\sinh\frac{\lambda_m b}{2} - \left[\frac{D}{\kappa^2 Gh} + \frac{1}{2}\left(\frac{a}{m\pi}\right)^2\right]\sinh\frac{m\pi b}{2a}\cosh\frac{\lambda_m b}{2} - \frac{ab}{4m\pi}\operatorname{sech}\frac{m\pi b}{2a}\cosh\frac{\lambda_m b}{2}\right\}}, \tag{B.2b}$$

$$C_{3m} = \frac{ab}{4m\pi}\tanh\frac{m\pi b}{2a}C_{2m} - \mu_m^+\operatorname{sech}\frac{m\pi b}{2a}, \tag{B.2c}$$

$$C_{4m} = \frac{ab}{4m\pi}\coth\frac{m\pi b}{2a}C_{1m} - \mu_m^-\operatorname{csch}\frac{m\pi b}{2a}, \tag{B.2d}$$

$$C_{5m} = -\frac{2}{1-v}\left(\frac{m\pi}{a\lambda_m}\right)\operatorname{sech}\frac{\lambda_m b}{2}\left[\cosh\frac{m\pi b}{2a}C_{2m} + \frac{\kappa^2 Gh}{D}\mu_m^+\right], \tag{B.2e}$$

$$C_{6m} = -\frac{2}{1-v}\left(\frac{m\pi}{a\lambda_m}\right)\operatorname{csch}\frac{\lambda_m b}{2}\left[\sinh\frac{m\pi b}{2a}C_{1m} + \frac{\kappa^2 Gh}{D}\mu_m^-\right], \tag{B.2f}$$

where

$$\mu_m^+ = \frac{\mathsf{M}_m^{\mathrm{K}}(b/2) + \mathsf{M}_m^{\mathrm{K}}(-b/2)}{2\kappa^2 Gh}, \qquad \mu_m^- = \frac{\mathsf{M}_m^{\mathrm{K}}(b/2) - \mathsf{M}_m^{\mathrm{K}}(-b/2)}{2\kappa^2 Gh}. \tag{B.3a, b}$$

B.2. SFSF Lévy plates

Consider the Lévy plate where the edges at $y = \pm b/2$ are free. The boundary conditions of these edges are

$$M_{yy}^{\mathrm{M}} = M_{yy}^{\mathrm{K}} = 0, \qquad Q_y^{\mathrm{M}} = V_y^{\mathrm{K}} = 0, \qquad M_{xy}^{\mathrm{M}} = 0, \tag{B.4a–c}$$

where $V_y^{\mathrm{K}} = Q_y^{\mathrm{K}} + \partial M_{xy}^{\mathrm{K}}/\partial x$ is the Kirchhoff effective shear force.

In view of Eqs. (16b), (16c), (16e), (20a)–(20c) and (B.4a–c), the constants are found to be

$$C_{1m} = \frac{\eta_m^+ \tanh\frac{m\pi b}{2a} + \xi_m^+\left\{\lambda_m \tanh\frac{\lambda_m b}{2} - \frac{a}{2m\pi}\left[\lambda_m^2 + \left(\frac{m\pi}{a}\right)^2\right]\tanh\frac{m\pi b}{2a}\right\}}{\left\{\left[\frac{3+v}{2(1-v)} + \left(\frac{m\pi}{a}\right)^2\frac{D}{\kappa^2 Gh}\right]\sinh\frac{m\pi b}{2a} - \lambda_m\left(\frac{m\pi}{a}\right)\frac{D}{\kappa^2 Gh}\cosh\frac{m\pi b}{2a}\tanh\frac{\lambda_m b}{2} + \frac{m\pi b}{4a}\operatorname{sech}\frac{m\pi b}{2a}\right\}}, \tag{B.5a}$$

$$C_{2m} = \frac{\eta_m^- \coth\frac{m\pi b}{2a} + \xi_m^-\left\{\lambda_m \coth\frac{\lambda_m b}{2} - \frac{a}{2m\pi}\left[\lambda_m^2 + \left(\frac{m\pi}{a}\right)^2\right]\coth\frac{m\pi b}{2a}\right\}}{\left\{\left[\frac{3+v}{2(1-v)} + \left(\frac{m\pi}{a}\right)^2\frac{D}{\kappa^2 Gh}\right]\cosh\frac{m\pi b}{2a} - \lambda_m\left(\frac{m\pi}{a}\right)\frac{D}{\kappa^2 Gh}\sinh\frac{m\pi b}{2a}\coth\frac{\lambda_m b}{2} - \frac{m\pi b}{4a}\operatorname{csch}\frac{m\pi b}{2a}\right\}}, \tag{B.5b}$$

$$\begin{aligned} C_{3m} &= \left\{\eta_m^- - \xi_m^-\left(\frac{a}{2m\pi}\right)\left[\lambda_m^2 + \left(\frac{m\pi}{a}\right)^2\right]\right\}\left(\frac{a}{m\pi}\right)^2\operatorname{csch}\frac{m\pi b}{2a} + \left(\frac{a}{m\pi}\right)\left[\frac{b}{4}\coth\frac{m\pi b}{2a}\right. \\ &\qquad \left. - \left(\frac{a}{m\pi}\right)\frac{1+v}{2(1-v)}\right]C_{2m} \\ &= -\xi_m^-\lambda_m\left(\frac{a}{m\pi}\right)^2\operatorname{sech}\frac{m\pi b}{2a}\coth\frac{\lambda_m b}{2} + \left[\frac{1}{1-v}\left(\frac{a}{m\pi}\right)^2 + \frac{D}{\kappa^2 Gh} + \frac{ab}{4m\pi}\tanh\frac{m\pi b}{2a}\right. \\ &\qquad \left. - \left(\frac{a\lambda_m}{m\pi}\right)\frac{D}{\kappa^2 Gh}\tanh\frac{m\pi b}{2a}\coth\frac{\lambda_m b}{2}\right]C_{2m}, \end{aligned} \tag{B.5c}$$

$$\begin{aligned} C_{4m} &= \left\{\eta_m^+ - \xi_m^+\left(\frac{a}{2m\pi}\right)\left[\lambda_m^2 + \left(\frac{m\pi}{a}\right)^2\right]\right\}\left(\frac{a}{m\pi}\right)^2\operatorname{sech}\frac{m\pi b}{2a} + \left(\frac{a}{m\pi}\right)\left[\frac{b}{4}\tanh\frac{m\pi b}{2a}\right. \\ &\qquad \left. - \left(\frac{a}{m\pi}\right)\frac{1+v}{2(1-v)}\right]C_{1m} \\ &= -\xi_m^+\lambda_m\left(\frac{a}{m\pi}\right)^2\operatorname{csch}\frac{m\pi b}{2a}\tanh\frac{\lambda_m b}{2} + \left[\frac{1}{1-v}\left(\frac{a}{m\pi}\right)^2 + \frac{D}{\kappa^2 Gh} + \frac{ab}{4m\pi}\coth\frac{m\pi b}{2a}\right. \\ &\qquad \left. - \left(\frac{a\lambda_m}{m\pi}\right)\frac{D}{\kappa^2 Gh}\coth\frac{m\pi b}{2a}\tanh\frac{\lambda_m b}{2}\right]C_{1m}, \end{aligned} \tag{B.5d}$$

$$C_{5m} = -\frac{2}{1-v}\operatorname{csch}\frac{\lambda_m b}{2}\left[\sinh\frac{m\pi b}{2a}C_{2m} + \frac{a}{m\pi}\left(\frac{\kappa^2 Gh}{D}\right)\xi_m^-\right], \tag{B.5e}$$

$$C_{6m} = -\frac{2}{1-v}\operatorname{sech}\frac{\lambda_m b}{2}\left[\cosh\frac{m\pi b}{2a}C_{1m} + \frac{a}{m\pi}\left(\frac{\kappa^2 Gh}{D}\right)\xi_m^+\right], \tag{B.5f}$$

where

$$\eta_m^+ = \frac{M_{mxy}^{\mathrm{K}}(b/2) + M_{mxy}^{\mathrm{K}}(-b/2)}{2D(1-\nu)}, \qquad \eta_m^- = \frac{M_{mxy}^{\mathrm{K}}(b/2) - M_{mxy}^{\mathrm{K}}(-b/2)}{2D(1-\nu)}, \tag{B.6a, b}$$

$$\xi_m^+ = \frac{Q_{my}^{\mathrm{K}}(b/2) + Q_{my}^{\mathrm{K}}(-b/2)}{2\kappa^2 Gh}, \qquad \xi_m^- = \frac{Q_{my}^{\mathrm{K}}(b/2) - Q_{my}^{\mathrm{K}}(-b/2)}{2\kappa^2 Gh}. \tag{B.6c, d}$$

B.3. SSSC Lévy plates

Next, consider the Lévy plate where the edges at $x = 0$, $x = a$ and $y = -b/2$ are simply supported while the edge at $y = b/2$ is clamped. The boundary conditions are

$$M_{yy}^{\mathrm{M}} = M_{yy}^{\mathrm{K}} = 0, \qquad w^{\mathrm{M}} = w^{\mathrm{K}} = 0, \qquad \phi_x = 0 \text{ for simply supported edge } y = -b/2 \tag{B.7a–c}$$

and

$$w^{\mathrm{M}} = w^{\mathrm{K}} = 0, \qquad \phi_y = \frac{\partial w^{\mathrm{K}}}{\partial y} = 0, \qquad \phi_x = 0 \text{ for clamped edge } y = b/2. \tag{B.7d–f}$$

In view of Eqs. (14)–(15b), (16b), (20a)–(20c), (B.7a–c) and (B.7d–f), one obtains

$$C_{1m} = \frac{\frac{m\pi}{a\lambda_m}\left(\mu_m^+ \tanh\frac{\lambda_m b}{2} + \mu_m^- \coth\frac{\lambda_m b}{2}\right) - \left(\mu_m^+ \tanh\frac{m\pi b}{2a} + \mu_m^- \coth\frac{m\pi b}{2a}\right)}{\left\{\frac{ab}{2m\pi}\sinh\frac{m\pi b}{2a} + \left[\frac{D}{\kappa^2 Gh} + \frac{1}{2}\left(\frac{m\pi}{a}\right)^2\right]\cosh\frac{m\pi b}{a}\operatorname{sech}\frac{m\pi b}{2a} - \frac{ab}{4m\pi}\left[\cosh^2\frac{m\pi b}{2a}\operatorname{csch}\frac{m\pi b}{2a} + \sinh^3\frac{m\pi b}{2a}\operatorname{sech}^2\frac{m\pi b}{2a}\right] - \frac{2m\pi}{a\lambda_m}\frac{D}{\kappa^2 Gh}\sinh\frac{m\pi b}{2a}\coth\lambda_m b\right\}}, \tag{B.8a}$$

$$C_{2m} = C_{1m}\tanh\frac{m\pi b}{2a}, \tag{B.8b}$$

$$C_{3m} = \text{right hand side of Eq. (B.2c)}, \tag{B.8c}$$

$$C_{4m} = \text{right hand side of Eq. (B.2d)}, \tag{B.8d}$$

$$C_{5m} = \text{right hand side of Eq. (B.2e)}, \tag{B.8e}$$

$$C_{6m} = \text{right hand side of Eq. (B.2f)}, \tag{B.8f}$$

where μ_m^+ and μ_m^- have the same meanings as in Eq. (B.3a,b)

B.4. SSSF Lévy plates

Finally, consider the Lévy plate with the edges at $x = 0$, $x = a$ and $y = -b/2$ simply supported and with the edge at $y = b/2$ free. The boundary conditions are given by

$$M_{yy}^{\mathrm{M}} = M_{yy}^{\mathrm{K}} = 0, \qquad w^{\mathrm{M}} = w^{\mathrm{K}} = 0, \qquad \phi_x = 0 \text{ for simply supported edge } y = -b/2 \tag{B.9a–c}$$

and

$$M_{yy}^{\mathrm{M}} = M_{yy}^{\mathrm{K}} = 0, \qquad Q_y^{\mathrm{M}} = V_y^{\mathrm{K}} = 0, \qquad M_{xy}^{\mathrm{M}} = 0 \text{ for free edge } y = b/2. \tag{B.9d–f}$$

In view of Eqs. (14), (15a), (16b), (16c), (16e), (20a)–(20c), (B.9a–c) and (B.9d–f), the constants are found to be

$$C_{1m} = 0, \tag{B.10a}$$

$$C_{2m} = 0, \tag{B.10b}$$

$$C_{3m} = -\lambda_m \left(\frac{a}{m\pi}\right)^2 \frac{Q^{\mathrm{K}}_{my}(b/2)}{2\kappa^2 Gh} \operatorname{sech}\frac{m\pi b}{2a} \tanh \lambda_m b, \tag{B.10c}$$

$$C_{4m} = C_{3m} \coth\frac{m\pi b}{2a}, \tag{B.10d}$$

$$C_{5m} = -\frac{2}{1-\nu}\left(\frac{a}{m\pi}\right)\frac{Q^{\mathrm{K}}_{my}(b/2)}{D} \sinh\frac{\lambda_m b}{2} \operatorname{sech}\lambda_m b, \tag{B.10e}$$

$$C_{6m} = C_{5m} \coth\frac{\lambda_m b}{2}. \tag{B.10f}$$

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